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Some 2-groups from the view of Hilbert-Poincaré polynomials of $K(2)^*(BG)$

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Abstract

In this note we analyze Morava K-theory rings of classifying spaces of some groups of order 32 via Hilbert-Poincaré polynomials.

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1 Preliminaries

Let $R = \oplus_i R_i$ be a $\mathbb{N}$-graded k-algebra, all R_i be vector spaces over a field k. Then the series $HP(R,t) = \sum_i dim_k R_i t^i$ is called the Hilbert-Poincaré series of R.

If R is generated by h homogeneous elements of positive degrees $d_1, \cdots, d_h$, then the sum of the Hilbert-Poincaré series is a rational fraction

$$HP(R,t) = \frac{Q(t)}{\prod_{i=1}^{h}(1-t^{d_i})},$$

where Q is a polynomial with integer coefficients.

In this paper we analyze the Morava K-theory rings $K(2)^*(BG)$ at 2 of some groups of order 32. In each example the ring $K(2)^*(BG)$ is presented as a quotient of a polynomial ring $K(s)^*[x_1, x_2, \cdots, x_m]$ by an ideal I generated by explicit polynomials. In this situation the naive definition of $HP(t)$ does not work as there are the elements of negative degree in the ring of coefficients $K(2)^*$, the graded field $F_2[v_2, v_2^{-1}]$, with $|v_2| = -6$. Let degree of $K(2)^*$ be zero. Then the relations ideal I is not homogeneous with respect to variables $x_1, \ldots, x_n$, hence the quotient ring is not graded (but filtered) with respect to cohomological degree. One can replace the ideal I by some homogeneous ideal to reduce the definition to graded algebra case and measure how strong is the Hilbert-Poincaré polynomial in distinguishing some $K(2)^*(BG)$'s.

We use the following definitions of [11]. Let I be an ideal of a polynomial ring $k[x_1, \cdots, x_n]$ over a field k, and let $>$ be a global monomial ordering. For instance

(i) Graded reverse lexicographical ordering $>_{dp}$ (also denoted by degrevlex):

$$\begin{aligned} x^\alpha >_{dp} x^\beta \Leftrightarrow\ & \deg x^\alpha > \deg x^\beta \\ & \text{or } \deg x^\alpha = \deg x^\beta \text{ and } \exists 1 \le i \le n, \\ & \alpha_n = \beta_n, \cdots, \alpha_{i+1} = \beta_{i+1},\ \alpha_i < \beta_i. \end{aligned}$$

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(ii) Graded lexicographical ordering $>_{Dp}$ (also denoted by deglex):

$$\begin{aligned} x^\alpha >_{Dp} x^\beta \Leftrightarrow\ & \deg x^\alpha > \deg x^\beta \\ & \text{or } \deg x^\alpha = \deg x^\beta \text{ and } \exists 1 \le i \le n, \\ & \alpha_1 = \beta_1, \cdots, \alpha_{i-1} = \beta_{i-1},\ \alpha_i > \beta_i. \end{aligned}$$

Given a vector $w = (w_1, \cdots, w_n)$ of integers, we define a weighted degree of x^α by

$$w \deg(x^\alpha) := w_1\alpha_1 + \cdots w_n\alpha_n,$$

that is, the variable x_i has degree w_i. For a polynomial $f = \sum_\alpha a_\alpha x^\alpha$ we define thee weighted degree,

$$w \deg(f) := max\{w \deg(x^\alpha) \mid a_\alpha \neq 0\}.$$

Using the weighted degree in (i), respectively in (ii), with all $w_i > 0$, instead of the usual degree, we obtain the weighted reverse lexicographical ordering, $w_p(w_1, \cdots, w_n)$, respectively the lexicographical ordering, $W_p(w_1, \cdots, w_n)$.

For $k[x_1, \cdots, x_n]/I$, a filtered algebra, the Hilbert-Poincaré series w.r.t $>$ is defined as follows. Replace I by its leading ideal $L(I)$ generated by leading terms of the Gröbner basis of I. Then the ring $k[x_1, \cdots, x_n]/L(I)$ is a graded ring. By definition

$$HP(t, k[x_1, \cdots, x_n]/I) = HP(t, k[x_1, \cdots, x_n]/L(I))\ \ w.r.t. > .$$

Below, we compute the Hirbert-Poincaré series $HP(t)$ for $K(2)^*(BG)$ for some groups G with $|G| = 32$. In particular, there are groups G such that the $K(2)$-Euler characteristics $\chi_{2,2}(G)$ are same but $HP(t)$ are different.

2 Hilbert-Poincaŕe polynomials

Note that in the case of a finite group G the ring $K(s)^*(BG)$ is finite dimensional vector space over $K(s)^*(pt)$, so that the Hilbert-Poincaré series we defined is the polynomial. There exist 51 non-isomorphic groups of order 32. In the monograph [8] these groups are numbered by $1, \cdots, 51$. Some of these groups are classical and named. We consider the following examples

$$G_{34} = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid \mathbf{a}^4 = \mathbf{b}^4 = \mathbf{c}^2 = [\mathbf{a}, \mathbf{b}] = 1, \mathbf{cac} = \mathbf{a}^{-1}, \mathbf{cbc} = \mathbf{b}^{-1}\rangle,$$
$$G_{35} = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid \mathbf{a}^4 = \mathbf{b}^4 = [\mathbf{a}, \mathbf{b}] = 1, \mathbf{c}^2 = \mathbf{a}^2, \mathbf{cac}^{-1} = \mathbf{a}^{-1}, \mathbf{cbc}^{-1} = \mathbf{b}^{-1}\rangle.$$
$$G_{36} = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid \mathbf{a}^4 = \mathbf{b}^4 = \mathbf{c}^2 = [\mathbf{b}, \mathbf{c}] = 1, \mathbf{a}^{-1}\mathbf{ba} = \mathbf{b}^{-1}, \mathbf{cac} = \mathbf{a}^{-1}\rangle,$$
$$G_{37} = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid \mathbf{a}^4 = \mathbf{c}^2 = \mathbf{d}^2 = [\mathbf{b}, \mathbf{c}] = 1, \mathbf{d} = [\mathbf{a}, \mathbf{c}], \mathbf{b}^2 = \mathbf{a}^2, \mathbf{bab}^{-1} = \mathbf{a}^{-1}\rangle,$$
$$G_{38} = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid \mathbf{a}^4 = \mathbf{b}^2 = \mathbf{c}^4 = [\mathbf{a}, \mathbf{b}] = 1, \mathbf{cac}^{-1} = \mathbf{ac}^2, \mathbf{cbc}^{-1} = \mathbf{a}^2\mathbf{b}\rangle,$$
$$G_{39} = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid \mathbf{a}^4 = \mathbf{b}^4 = \mathbf{c}^2 = [\mathbf{a}, \mathbf{b}] = 1, \mathbf{cac} = \mathbf{a}^3, \mathbf{cbc} = \mathbf{a}^2\mathbf{b}^3\rangle,$$
$$G_{40} = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid \mathbf{a}^4 = \mathbf{b}^4 = 1, \mathbf{c}^2 = \mathbf{b}^2, [\mathbf{a}, \mathbf{b}] = 1, \mathbf{c}^{-1}\mathbf{ac} = \mathbf{a}^3, \mathbf{c}^{-1}\mathbf{bc} = \mathbf{a}^2\mathbf{b}^3\rangle,$$
$$G_{41} = \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid \mathbf{a}^4 = \mathbf{b}^4 = \mathbf{c}^2 = [\mathbf{a}, \mathbf{b}] = 1, \mathbf{cac} = \mathbf{a}^3\mathbf{b}^2, \mathbf{cbc} = \mathbf{a}^2\mathbf{b}\rangle,$$
$$D = \langle \mathbf{a}, \mathbf{b}, \mid \mathbf{a}^{16} = 1, \mathbf{b}^2 = 1, \mathbf{bab}^{-1} = \mathbf{a}^{-1}\rangle, \text{ the dihedral group,}$$
$$Q = \langle \mathbf{a}, \mathbf{b}, \mid \mathbf{a}^{16} = 1, \mathbf{b}^2 = \mathbf{a}^8, \mathbf{bab}^{-1} = \mathbf{a}^{-1}\rangle, \text{ the generalized quaternion group,}$$
$$SD = \langle \mathbf{a}, \mathbf{b}, \mid \mathbf{a}^{16} = 1, \mathbf{b}^2 = 1, \mathbf{bab}^{-1} = \mathbf{a}^7\rangle, \text{ the semi-dihedral group,}$$
$$QD = \langle \mathbf{a}, \mathbf{b}, \mid \mathbf{a}^{16} = \mathbf{b}^2 = 1, \mathbf{bab}^{-1} = \mathbf{a}^9\rangle, \text{ the quasi-dihedral group.}$$

Explicit calculations of Morava $K(s)^*(BG)$ rings for the groups of order 32 are scattered in the literature (see the references [1–7], [9–14]). For ease of presentation we set $s = 2$, $v_2 = v$ and discuss $K(2)^*(BG)$ for the twelve groups defined above. It is proved in [4], [5] that for the groups $G_{34}, \cdots G_{41}$ the ring $K(s)^*(BG)$ is the quotient of a polynomial ring in 6 variables over the field $K(s)^*(pt) = F_2[v_s, v_s^{-1}]$ by an ideal generated by 16 explicit polynomials (see Proposition 2.1).

Let $\chi_{s,2}$ be $K(s)^*$-Euler-characteristic of G, the difference between the $K(s)^*$-ranks of the even-dimensional and the odd-dimensional part of $K(s)^*(BG)$. It is proved in [10] that $K(s)^{odd}(BG)$ turns out to be trivial for all groups of order 32. It follows $\chi_{s,2} = rank_{K(s)^*}(K(s)^*(BG))$ for the groups we are considering. The latter can be calculated [9] in terms of abelian subgroups of G and Möbius function as follows

$$\chi_{s,2} = \sum_{A<G} \frac{|A|}{|G|}\mu_G(A)\chi_{s,2}(A), \tag{1}$$

where the sum is over all abelian subgroups $A < G$ and μ_G is a Möbius function defined recursively by

$$\sum_{A<A'} \mu_G(A') = 1,$$

where the sum is over all abelian subgroups $A' < G$ which contain A. (In particular, $\mu_G(A) = 1$ when A is maximal.)

For our examples these calculations are known by Brunetti [7] and Schuster [10]

$$\chi_{2,2}(G) = \begin{cases} 184 \text{ for } G = G_{34}, \cdots, G_{41}, \\ 142 \text{ for } G = D, Q, SD, \\ 352 \text{ for } G = QD. \end{cases} \tag{2}$$

One can define the Poincaré polynomial $P(t)$ of G by using the cyclical gradings of $K(s)^*$-modules: recall the Morava K-theory is a periodic cohomology theory, the coefficient ring $K(s)^*(pt)$ is the graded field $F_2[v_s, v_s^{-1}]$, with $|v_s| = -2(2^s - 1)$. Therefore as $K(s)^*$-module $K(s)^*(BG)$ is periodically graded. For $s = 2$, the unit v_2 is of degree -6 and assigning degrees 0, 2, or 4 to each element of $K(2)^*(BG)$, the Poincaré polynomial of finite group G is defined as

$$P(t) = 1 + c_0t^0 + c_2t^2 + c_4t^4, \tag{3}$$

where $c_i = rank_{k(2)^*} K(2)^i \mod 6$.

One immediate consequence of our results (the corollaries 2.4, 2.5 and 2.6) is the following

$$P(t) = \begin{cases} 1 + 61(1 + t^2 + t^4) \text{ for } G = G_{34}, \cdots, G_{41}, \\ 1 + 47(1 + t^2 + t^4) \text{ for } G = D, Q, SD, \\ 1 + 117(1 + t^2 + t^4) \text{ for } G = QD. \end{cases} \tag{4}$$

Actually (4) is easily predicted: it says that the for the groups under consideration the elements of $K(2)^*(BG)$ are equally distributed in dimensions 0, 2, 4 w.r.t the cyclic grading. Surely this was already known to N. Yagita. On the other hand the sum of all coefficients equals $\chi_{2,2}$ in (2).

One can conclude that naive Poincaré series $P(t)$ w.r.t cyclic grading of $K(2)^*(BG)$ can't see the difference between the groups in (4) with equal $K(2)^*$-Euler characteristics. Therefore we will ignore the cyclic grading.

We are particularly interesting in issues related to the comparison of the ring structures. Namely, suppose that for groups G_1 and G_2 the corresponding Morava rings are presented in terms of generators and relations as

$$R_i \equiv K(s)^*(BG_i) = K(s)^*[x_1, \cdots, x_n]/I_i.$$

Then one can ask how "close" the ring structures in R_1 and R_2 are? To see the difference in the ring structures we need the Hilbert-Poincaré series defined above.

For instance in Corollary 2.6 we fix a monomial ordering of the generators of Morava rings and compute the corresponding series $HP(t)$ for the groups G_{34}, G_{35}, G_{39}, G_{40} and G_{36}, G_{37}, G_{38}, G_{41} respectively.

We use SINGULAR code hilb [11]. It computes the Hilbert series $Q(t)$ and the Hilbert-Poincaré polynomials $HP(t) = Q(t)/\prod(1 - t^{w_i})$ defined above. If a weight vector w is given, then the Hilbert-Poincaré series is computed w.r.t. these weights w (by default all weights are set to 1). In computation through SINGULAR we can put $v_2 = 1$ and halve the degrees of all $K(2)^*$-generators.

Morava $K(2)^*(BG)$-rings for the groups under consideration are as follows.

Proposition 2.1. Let G_i be one of the groups $G_{34}, \cdots, G_{41}$. Then

$$K(2)^*(BG_i) \cong K(2)^*[a, b, c, x_1, x_2, y_1, y_2, T]/I_i,$$

where $|a| = |b| = |c| = |x_1| = |y_1| = 2$, $|x_2| = |y_2| = |T| = 4$ and the relations ideal I_i is as follows

$I_{34} = (a^4, b^4, c^4, c + x_1 + v{x_2}^2 + v^3{x_1}^2{x_2}^4, y_1 + c + v{y_2}^2 + v^3{y_1}^2{y_2}^4, c(c + x_1 + vc^2x_2), c(c + y_1 + vc^2y_2), a(a + x_1 + va^2x_2), b(b + y_1 + vb^2y_2), v^2{y_2}^4 + b^2 + bc, v^2{x_2}^4 + a^2 + ac, (c + x_1 + vc^2x_2)(b + y_1 + vb^2y_2) + vb^3T, (c + y_1 + vc^2y_2)(a + x_1 + va^2x_2) + va^3T, T^2 + Tx_1y_1 + x_2y_1(c + y_1 + vc^2y_2) + x_1y_2(c + x_1 + vc^2x_2), T(a + x_1 + va^2x_2) + va^3x_2(c + y_1), T(b + y_1 + vb^2y_2) + vb^3y_2(c + x_1), cT)$;

$I_{35} = (a^4, b^4, c^4, c + x_1 + v{x_2}^2 + v^3{x_1}^2{x_2}^4, y_1 + v{y_2}^2 + v^3{y_1}^2{y_2}^4, c(c + x_1 + vc^2x_2), c(c + y_1 + vc^2y_2), a(a + x_1 + va^2x_2), b(b + y_1 + vb^2y_2), v^2{y_2}^4 + b^2 + bc + c^2, v^2{x_2}^4 + a^2 + ac, (c + x_1 + vc^2x_2)(b + y_1 + vb^2y_2) + vb^3T, (c + y_1 + vc^2y_2)(a + x_1 + va^2x_2) + va^3T, T^2 + Tx_1y_1 + x_2y_1(c + y_1 + vc^2y_2) + x_1y_2(c + x_1 + vc^2x_2), T(a + x_1 + va^2x_2) + va^3x_2(c + y_1), T(b + y_1 + vb^2y_2) + vb^3y_2(c + x_1), cT)$;

$I_{36} = (a^4, b^4, c^4, x_1 + v{x_2}^2 + v^3{x_1}^2{x_2}^4 + b, y_1 + v{y_2}^2 + v^3{y_1}^2{y_2}^4 + c, c(c + x_1 + vc^2x_2), c(c + y_1 + vc^2y_2), a(a + y_1 + va^2y_2), b(b + x_1 + vb^2x_2), v^2{y_2}^4 + a^2 + ac, v^2{x_2}^4 + c^2 + bc, (c + x_1 + vc^2x_2)(a + y_1 + va^2y_2) + va^3T, (c + y_1 + vc^2y_2)(b + x_1 + vb^2x_2) + vb^3T, T^2 + Tx_1y_1 + x_2y_1(c + y_1 + vc^2y_2) + x_1y_2(c + x_1 + vc^2x_2), T(b + x_1 + vb^2x_2) + vb^3x_2(c + y_1), T(a + y_1 + va^2y_2) + va^3y_2(c + x_1), cT)$;

$I_{37} = (a^4, b^4, c^4, x_1 + v{x_2}^2 + v^3{x_1}^2{x_2}^4 + b + c + vb^2c^2, y_1 + v{y_2}^2 + v^3{y_1}^2{y_2}^4, c(c + x_1 + vc^2x_2), c(c + y_1 + vc^2y_2), a(a + y_1 + va^2y_2), b(b + x_1 + vb^2x_2), v^2{y_2}^4 + a^2 + ac + c^2, v^2{x_2}^4 + bc, (c + x_1 + vc^2x_2)(a + y_1 + va^2y_2) + va^3T, (c + y_1 + vc^2y_2)(b + x_1 + vb^2x_2) + vb^3T, T^2 + Tx_1y_1 + x_2y_1(c + y_1 + vc^2y_2) + x_1y_2(c + x_1 + vc^2x_2), T(b + x_1 + vb^2x_2) + vb^3x_2(c + y_1), T(a + y_1 + va^2y_2) + va^3y_2(c + x_1), cT)$;

$I_{38} = (a^4, b^4, c^4, x_1 + v{x_2}^2 + v^3{x_1}^2{x_2}^4 + a, y_1 + v{y_2}^2 + v^3{y_1}^2{y_2}^4 + a + b + c + va^2b^2 + vb^2c^2 + va^2c^2, c(c + x_1 + vc^2x_2), c(c + y_1 + vc^2y_2), a(a + x_1 + va^2x_2), b(b + y_1 + vb^2y_2), v^2{y_2}^4 + a^2 + bc + vabc^3, v^2{x_2}^4 + c^2 + ac, (c + x_1 + vc^2x_2)(b + y_1 + vb^2y_2) + vb^3T, (c + y_1 + vc^2y_2)(a + x_1 + va^2x_2) + va^3T, T^2 + Tx_1y_1 + x_2y_1(c + y_1 + vc^2y_2) + x_1y_2(c + x_1 + vc^2x_2), T(a + x_1 + va^2x_2) + va^3x_2(c + y_1), T(b + y_1 + vb^2y_2) + vb^3y_2(c + x_1), cT)$;

$I_{39} = (a^4, b^4, c^4, x_1 + v{x_2}^2 + v^3{x_1}^2{x_2}^4 + b + c + vb^2c^2, y_1 + v{y_2}^2 + v^3{y_1}^2{y_2}^4 + c, c(c + x_1 + vc^2x_2), c(c + y_1 + vc^2y_2), a(a + x_1 + va^2x_2), b(b + y_1 + vb^2y_2), v^2{y_2}^4 + b^2 + bc, v^2{x_2}^4 + a^2 + b^2 + ac + vabc^3, (c + x_1 + vc^2x_2)(b + y_1 + vb^2y_2) + vb^3T, (c + y_1 + vc^2y_2)(a + x_1 + va^2x_2) + va^3T, T^2 + Tx_1y_1 + x_2y_1(c + y_1 + vc^2y_2) + x_1y_2(c + x_1 + vc^2x_2), T(a + x_1 + va^2x_2) + va^3x_2(c + y_1), T(b + y_1 + vb^2y_2) + vb^3y_2(c + x_1), cT)$;

$I_{40} = (a^4, b^4, c^4, x_1 + v{x_2}^2 + v^3{x_1}^2{x_2}^4 + b + c + vb^2c^2, y_1 + v{y_2}^2 + v^3{y_1}^2{y_2}^4, c(c + x_1 + vc^2x_2), c(c + y_1 + vc^2y_2), a(a + x_1 + va^2x_2), b(b + y_1 + vb^2y_2), v^2{y_2}^4 + b^2 + c^2 + bc, v^2{x_2}^4 + a^2 + b^2 + ac + vabc^3, (c + x_1 + vc^2x_2)(b + y_1 + vb^2y_2) + vb^3T, (c + y_1 + vc^2y_2)(a + x_1 + va^2x_2) + va^3T, T^2 + Tx_1y_1 + x_2y_1(c + y_1 + vc^2y_2) + x_1y_2(c + x_1 + vc^2x_2), T(a + x_1 + va^2x_2) + va^3x_2(c + y_1), T(b + y_1 + vb^2y_2) + vb^3y_2(c + x_1), cT)$;

$I_{41} = (a^4, b^4, c^4, x_1 + v{x_2}^2 + v^3{x_1}^2{x_2}^4 + b + c + vb^2c^2, y_1 + v{y_2}^2 + v^3{y_1}^2{y_2}^4 + a + b + c + va^2b^2 + vb^2c^2 + va^2c^2, c(c + x_1 + vc^2x_2), c(c + y_1 + vc^2y_2), a(a + x_1 + va^2x_2), b(b + y_1 + vb^2y_2), v^2{y_2}^4 + a^2 + bc + vabc^3, v^2{x_2}^4 + a^2 + b^2 + ac + vabc^3, (c + x_1 + vc^2x_2)(b + y_1 + vb^2y_2) + vb^3T, (c + y_1 + vc^2y_2)(a + x_1 + va^2x_2) + va^3T, T^2 + Tx_1y_1 + x_2y_1(c + y_1 + vc^2y_2) + x_1y_2(c + x_1 + vc^2x_2), T(a + x_1 + va^2x_2) + va^3x_2(c + y_1), T(b + y_1 + vb^2y_2) + vb^3y_2(c + x_1), cT)$;

Morava rings $K(s)^*(BG)$ for the groups D, Q, QD, SD are calculated in [6, 2]. In particular, one has

Proposition 2.2. Let G be one of the groups D, Q, SD. Then $K(2)^*(BG) \cong K(2)^*[c, x, c_2]/I_G$, where $|c| = |x| = 2$, $|c_2| = 4$ and the relations ideal I_G is as follows

$$\begin{aligned} I_D =&(c^4, x^4, vcc_2{}^2 + vc^3c_2, v^{42}c_2{}^{64} + cx + x^2, vxc_2{}^2 + vxc^2c_2 + v^{84}c_2{}^{127}+ \\ &v^{82}c_2{}^{124} + v^{78}c_2{}^{118} + v^{70}c_2{}^{106} + v^{54}c_2{}^{82} + v^{22}c_2{}^{34}); \\ I_Q =&(c^4, x^4, vcc_2{}^2 + vc^3c_2 + c^2, v^{42}c_2{}^{64} + cx + x^2, vxc_2{}^2 + vxc^2c_2 + v^{84}c_2{}^{127}+ \\ &v^{82}c_2{}^{124} + v^{78}c_2{}^{118} + v^{70}c_2{}^{106} + v^{54}c_2{}^{82} + v^{22}c_2{}^{34} + cx); \\ I_{SD} =&(c^4, x^4, vcc_2{}^2 + vc^3c_2 + cx, v^{42}c_2{}^{64} + cx + x^2, vxc_2{}^2 + vxc^2c_2 + v^{84}c_2{}^{127}+ \\ &v^{82}c_2{}^{124} + v^{78}c_2{}^{118} + v^{70}c_2{}^{106} + v^{54}c_2{}^{82} + v^{22}c_2{}^{34} + cx). \end{aligned}$$

Proposition 2.3. Let QD be the quasi-dihedral group of order 32 as above. Then $K(2)^*(BG) \cong K(2)^*[x, y, c_1, c_2]/I_{QD}$, where $|x| = |y| = |c_1| = 2$, $|c_2| = 4$ and the relations ideal I_{QD} is

$$(x^4, y^4, x(c_1 + x + vx^2c_2), y(c_1 + y + vy^2c_2), (c_1 + x + vx^2c_2)(c_1 + y + vy^2c_2), x + v^{21}c_2{}^{32}).$$

Corollary 2.4. For $G = D, Q, SD$ the Hilbert first series $Q(t)$ and the Hilbert-Poincaré Series $HP(t)$ of $K(2)^*(BG)$ w.r.t. w_p ordering $(c, x, z)(1, 1, 2)$ is equal to

$$\begin{aligned} Q(t) =&1 - t^3 - 2t^4 + t^6 + t^7 + t^8 - t^{10} - t^{72} + 2t^{71} - t^{70} - t^{68} + 3t^{66} - 2t^{65} = \\ &(1-t)^3(1+t)(1 + 2t + 4t^2 + 5t^3 + 5t^4 + 4t^5 + 3t^6 + \sum_{i=7}^{64} 2t^i + t^{66} + t^{68}); \\ HP(t) =&Q(t)/(1-t)^2(1-t^2) = \\ &1 + 2t + 4t^2 + 5t^3 + 5t^4 + 4t^5 + 3t^6 + \sum_{i=7}^{64} 2t^i + t^{66} + t^{68}. \end{aligned}$$

One can read off the $K(2)^*$ basis of $K(2)^*(BG)$ w.r.t the ordering we fixed. See for instance [6] page 3712. Again, we can put $v_2 = 1$ and halve the degrees. Then by counting the degrees of basis elements one can compute the coefficients of $HP(t)$.

Instead we used the singular codes to compute $HP(t)$ and at the same time checked the calculations in [6].

Similarly one has for $K(2)^*(BQD)$.

Corollary 2.5. The Hilbert first series $Q(t)$ and the Hilbert-Poincaré Series $HP(t)$ of $K(2)^*(BQD)$ w.r.t. w_p ordering $(c_1, x, y, c_2)(1, 1, 1, 2)$ is equal to

$$\begin{aligned} Q(t) =&(t-1)^4(t^{32}+1)(t^{16}+1)(t^8+1)(t^4+1)(t^2+1)(t^4+3t^3+3t^2+3t+1)(t+1); \\ HP(t) =&Q(t)/(1-t)^3(1-t^2) = \\ &(1+t^{32})(1+t^{16})(1+t^8)(1+t^4)(1+t^2)(1+3t+3t^2+3t^3+t^4). \end{aligned}$$

Consider now the groups $G_{34}, \cdots, G_{41}$. All eight groups have $K(2)^*$-Euler characteristic 184.

Corollary 2.6. The Hilbert first series $Q(t)$ and the Hilbert-Poincaré Series $HP(t)$ of $K(2)^*(BG)$, for the groups $G_{34}, \cdots, G_{41}$, w.r.t. w_p ordering

$$(a, b, c, y_1, x_1, y_2, x_2, T), (1, 1, 1, 1, 1, 2, 2, 2),$$

are given by the following table

$G_{34}, G_{35}, G_{39}, G_{40}$:
$Q(t) = (1-t)^8(1+t)^3(1+t^2)(1+5t+13t^2+19t^3+21t^4+16t^5+11t^6+5t^7+t^8)$;
$HP(t) = 1+5t+14t^2+24t^3+34t^4+35t^5+32t^6+21t^7+12t^8+5t^9+t^{10}$;

$G_{36}, G_{37}, G_{38}, G_{41}$:
$(1-t)^8(1+t)^3(1+5t+14t^2+25t^3+34t^4+35t^5+31t^6+21t^7+12t^8+5t^9+t^{10})$;
$1+5t+14t^2+25t^3+34t^4+35t^5+31t^6+21t^7+12t^8+5t^9+t^{10}$.

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